

The self-consistent microscopic model of an energy spectrum of a superfluidity ^4He with the Hermitian form of Bogoliubov-Zubarev Hamiltonian.

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(Dated: October 8, 2010)

Based on the collective variables representation with the Hermitian form of Bogoliubov-Zubarev Hamiltonian the self-consistent oscillator model of a ground state and excited states of Bose liquid has been proposed. The new method of calculation of anharmonic terms in this Hamiltonian and its interpretation have been presented. The dispersion equation for a collective excitation in a superfluid ^4He has been obtained in self-consistent way, where real and virtual processes of decay of a collective excitation were considered. The end point, determined by a threshold of collective excitation's decay on two rotons, of the dispersion curve has been obtained and it was shown that the dispersion curve strongly depends on property of its stability. An approach with a structure factor has been realized without using of any adaption parameters. Based on the oscillator model the new method of self-consistent calculation of a ground state energy and density of Bose condensate has been proposed. The model of suppression of Bose condensate has been presented.

PACS numbers: 67.25.D-, 67.25.dt, 67.10.Ba

I. INTRODUCTION

Up to present time many various models have been proposed for microscopic description of a superfluid helium. Most of the models are based on two approaches. The first one is the theoretical-field formalism. Its characteristic feature is explicit application of Bose-condensate (BC) and introduction of the higher BC in some models [1–4]. The second one is the quantum-mechanical approach, where the Schrodinger equation is solved for a ground state and for lowest excited states of N interacting particles. This approach originates from the papers of Feynman and Cohen [5] where connection between the structure factor and the dispersion curve of a collective excitation (CE) was obtained, however this connection is correct at small values of a wave vector only ($k \rightarrow 0$). The quantum-mechanical approach does not use BC explicitly unlike the theoretical-field approach but it calculates BC knowing the wave function of a ground state.

The quantum-mechanical approach has been developed essentially due to use of Bogoliubov-Zubarev formalism [6]. In this representation Hamiltonian of a bosons' system is written in the terms $\rho_{\mathbf{k}}$ and $\partial/\partial\rho_{\mathbf{k}}$ where $\rho_{\mathbf{k}}$ is a Fourier transform of fluctuation of density. As it has been shown in the papers [7, 8] normal motion and superfluidity motion can be separated in the N -particle Schrodinger equation. The oscillations of Bose liquid is obtained in the harmonic approximation. Zero-point oscillations correspond to a ground state of liquid, and excited states - to any collective excitations: phonons, rotons, maxsons and so on. However the non-harmonic terms of the Hamiltonian play a essential role. The first correction to the harmonic approximation has been calculated in above-mentioned papers, the second correction has been calculated in the paper [9]. Unfortunately any small parameters aren't in this expansion and calculation of each new correction is accompanied by large mathematical difficulties. Contribution of the anharmonic terms in energy of a liquid has been evaluated by Brillouin-Wigner perturbation procedure in the papers [10, 11], the method of Green function has been used [12], the "shadow wave function" approach has been developed [13].

The dispersion curve of CE obtained by some authors [5, 9–11] has satisfactory coincidence with the experimental spectrum if value of the wave vector is $k < 2.5\text{A}^{-1}$. In the region $k > 3\text{A}^{-1}$ the experimental spectrum reaches the "shelf", that is hybridization of CE with the two-rotion level is observed [14]. The region $k > 3\text{A}^{-1}$ has been investigated by the theoretical-field method in [15], where it has been shown that the "shelf" on the dispersion curve caused by decay of CE into two rotions and by the end of the spectrum. Calculation of the next corrections to the spectrum of CE doesn't give the end point of a dispersion curve. We have to note that the potential of interaction between atoms is known badly on small distances $r < 2.5\text{A}$. A ground state energy is very sensitive to interaction on small distances just. The situation is complicated by fact that He II is liquid however complete theory of liquid

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doesn't exist till now. That's why authors proceed from gas representations or the models of quantum crystal [16–18].

Bogoliubov-Zubarev Hamiltonian is non-Hermitian because the transition from Cartesian coordinates to the collective coordinates is non-unitary. This Hamiltonian has been used in the many aforesaid paper [6–9, 12, 19–22]. But the non-hermiticity must lead to violation of some theorems of quantum mechanics. However an operator of speed $g_{\mathbf{k}}$ canonical conjugated with $\rho_{\mathbf{k}}$ has been introduced in the paper of Sunakawa [10], and his Hamiltonian represented by these terms is Hermitian. The Hermitian form of Bogoliubov-Zubarev Hamiltonian and the Jacobian of transition to collective coordinates have been obtained in the book [23]. However the anharmonic terms of this Hamiltonian has not been calculated.

In the Section II superfluid motion and normal motion have been separated in the Schrodinger equation with the Hermitian form of Bogoliubov-Zubarev Hamiltonian. In the Section III the oscillator model of Bose liquid has been formulated in the Random Phase Approximation (RPA). In the Section IV contribution of anharmonic terms of the Hamiltonian in the spectrum of CE has been calculated. It has been shown that this correction describes decay of CE and the dispersion curve has the end point k_C . In the Section V a ground state energy E_0 has been calculated taking into account anharmonic terms in the Hamiltonian with help the oscillator model. The mechanism of suppression of BC has been described and density of BC at zero temperature has been calculated.

II. THE EQUATION OF MOTION AND THE SPACE OF COLLECTIVE VARIABLES

Let's consider N interacting Bose particles of mass m confined in a macroscopic volume V . Hamiltonian of the system has a form:

$$\hat{H} = \sum_{1 \leq j \leq N} \frac{\hat{p}^2}{2m} + \sum_{1 \leq i < j \leq N} \Phi(|\mathbf{r}_i - \mathbf{r}_j|), \quad (1)$$

where \hat{p} is the operator of momentum of a particle, the operator $\Phi(|\mathbf{r}_i - \mathbf{r}_j|)$ is an energy of interaction of two particles. The waves functions must be symmetrical for any rearrangements of coordinates of any pairs from N particles and it describes oscillations of Bose liquid. The space of the collective variables is the suitable multitude of variables describing collective motion of a system [24, 25]. Collective variables are Fourier transform of fluctuation of density $\Delta n(\mathbf{r}) = \sum_{j=1}^N \delta(\mathbf{r} - \mathbf{r}_j) - N/V$:

$$\rho_{\mathbf{k}} = \frac{1}{\sqrt{N}} \sum_{j=1}^N \exp(-i\mathbf{k}\mathbf{r}) = \rho_{\mathbf{k}}^c - i\rho_{\mathbf{k}}^s, \quad (2)$$

where

$$\rho_{\mathbf{k}}^c = \frac{1}{\sqrt{N}} \sum_{j=1}^N \cos(\mathbf{k}\mathbf{r}), \quad \rho_{\mathbf{k}}^s = \frac{1}{\sqrt{N}} \sum_{j=1}^N \sin(\mathbf{k}\mathbf{r}). \quad (3)$$

And what's more the correlations have a place:

$$\rho_{\mathbf{k}}^* = \rho_{-\mathbf{k}} \quad \Rightarrow \quad \rho_{\mathbf{k}}^c = \rho_{-\mathbf{k}}^c, \quad \rho_{\mathbf{k}}^s = -\rho_{-\mathbf{k}}^s. \quad (4)$$

This means, that it is necessary to consider the values $\rho_{\mathbf{k}}$ with indexes \mathbf{k} from the half-space of their possible values only.

The transition from the Cartesian coordinates $(\mathbf{r}_1, \dots, \mathbf{r}_N)$ to the variables $\rho_{\mathbf{k}}$ is nonunitary because quantity of the Cartesian coordinates is $3N$ but quantity of $\rho_{\mathbf{k}}$ -variables is infinity. Hence superfluous variables exist among collective variables. The transition to $\rho_{\mathbf{k}}$ -representation must be done with the Jacobian which equalizes a volume of the configuration space $\int d\mathbf{r}_1 \dots \int d\mathbf{r}_N = V^N$ to a volume of the $\rho_{\mathbf{k}}$ -space:

$$V^N = \prod'_{\mathbf{k} \neq 0} \int_{-\sqrt{N}}^{\sqrt{N}} d\rho_{\mathbf{k}}^c \int_{-\sqrt{N}}^{\sqrt{N}} d\rho_{\mathbf{k}}^s J, \quad (5)$$

where the prime at the symbol of multiplication means that variables \mathbf{k} are taken from the half-space only. In a representation of the new variables (2) the Hamiltonian (1) has the form:

$$\hat{H}_{\text{BZ}} = \sum_{\mathbf{k}_1 \neq 0} \varepsilon(k_1) \left(\rho_{\mathbf{k}_1} \frac{\partial}{\partial \rho_{\mathbf{k}_1}} - \frac{\partial^2}{\partial \rho_{\mathbf{k}_1} \partial \rho_{-\mathbf{k}_1}} \right) + \sum_{\mathbf{k}_1 \neq 0} \sum_{\mathbf{k}_2 \neq 0} \frac{\varepsilon(\mathbf{k}_1, \mathbf{k}_2)}{\sqrt{N}} \rho_{\mathbf{k}_1 + \mathbf{k}_2} \frac{\partial^2}{\partial \rho_{\mathbf{k}_1} \partial \rho_{\mathbf{k}_2}}$$

$$+\frac{N^2}{2V}\nu(0)+\frac{N}{2V}\sum_{\mathbf{k}_1\neq 0}\nu(k_1)\left(\rho_{\mathbf{k}_1}\rho_{-\mathbf{k}_1}-1\right), \quad (6)$$

and its name is Bogoliubov-Zubarev Hamiltonian. However this operator is non-Hermitian - the first term in the first brackets: $\rho_{\mathbf{k}_1}\frac{\partial}{\partial\rho_{\mathbf{k}_1}}$. This property of \hat{H}_{BZ} is caused by the the nonunitarian transition from Cartesian variables to collective variables $\rho_{\mathbf{k}}$.

In the paper [23] the Hermitian form of the Hamiltonian (6) has been obtained. Let the system is described by the wave function ψ which is normalized in $\rho_{\mathbf{k}}$ -representation:

$$\prod'_{\mathbf{k}\neq 0}\int_{-\sqrt{N}}^{\sqrt{N}}d\rho_{\mathbf{k}}^c\int_{-\sqrt{N}}^{\sqrt{N}}d\rho_{\mathbf{k}}^s J|\psi|^2=1. \quad (7)$$

Let's introduce the wave functions normalized without the Jacobian J :

$$\bar{\psi}=\psi\sqrt{J}. \quad (8)$$

Then Schrodinger equation is written as follows:

$$\hat{H}\bar{\psi}=E\bar{\psi}, \quad \hat{H}=J^{1/2}\hat{H}_{\text{BZ}}J^{-1/2}, \quad (9)$$

where the new Hamiltonian \hat{H} must be Hermitian. Proceeding from this condition we can find the Jacobian J . Then we have:

$$\begin{aligned} \hat{H} = & \sum_{\mathbf{k}_1\neq 0}\varepsilon(k_1)\left(-\frac{\partial^2}{\partial\rho_{\mathbf{k}_1}\partial\rho_{-\mathbf{k}_1}}-\frac{1}{4}\rho_{\mathbf{k}_1}\frac{\partial\ln J}{\partial\rho_{\mathbf{k}_1}}-\frac{1}{2}\right)+ \\ & \sum_{\mathbf{k}_1\neq 0}\sum_{\mathbf{k}_2\neq 0}\frac{\varepsilon(\mathbf{k}_1, \mathbf{k}_2)}{\sqrt{N}}\rho_{\mathbf{k}_1+\mathbf{k}_2}\frac{\partial^2}{\partial\rho_{\mathbf{k}_1}\partial\rho_{\mathbf{k}_2}}+\frac{N^2}{2V}\nu(0)+\frac{N}{2V}\sum_{\mathbf{k}_1\neq 0}\nu(k_1)\left(\rho_{\mathbf{k}_1}\rho_{-\mathbf{k}_1}-1\right), \end{aligned} \quad (10)$$

where the Jacobian must be found from the equation:

$$\rho_{\mathbf{k}_1}+\frac{\partial\ln J}{\partial\rho_{-\mathbf{k}_1}}-\frac{1}{\sqrt{N}}\sum_{\mathbf{k}_1\neq 0}^{\mathbf{k}_1+\mathbf{k}_2\neq 0}\frac{\mathbf{k}_1\mathbf{k}_2}{k_1^2}\rho_{\mathbf{k}_1+\mathbf{k}_2}\frac{\partial\ln J}{\partial\rho_{\mathbf{k}_2}}=0. \quad (11)$$

Solution of this equation can be written as

$$\ln J=\ln C+\sum_{n\geq 2}\frac{(-1)^{n-1}}{n(n-1)(\sqrt{N})^{n-2}}\sum_{\mathbf{q}_1\neq 0}\dots\sum_{\mathbf{q}_n\neq 0}\rho_{\mathbf{q}_1}\dots\rho_{\mathbf{q}_n}. \quad (12)$$

The constant C can be found from the condition (5). In our comprehension the equation (9) with the Hamiltonian (10) is *motion equation* in $\rho_{\mathbf{k}}$ -space. The equation (11) is *constraint equation* in this space. In Cartesian coordinate we have a discrete system from N particles but in collective coordinates the system is regarded as continuum. The constraints equation removes the superfluous degrees of freedom.

Normal motion and superfluid motion can be separated in the Schrodinger equation [7]. The wave function of a ground state describes zero-point oscillations of Bose liquid and has an exponential form e^U . The whole wave function (with excited states) has a form:

$$\bar{\psi}=e^U\varphi. \quad (13)$$

Then let's rewrite an energy of a liquid as

$$E=E_0+E-E_0\equiv E_0+E_{ext}, \quad (14)$$

where E_0 - is a ground state energy (energy of superfluid motion with wave function e^U), E_{ext} - is the energy of excitation (energy of normal motion with wave function φ). Then the equation (9) divides into the set of equations describing superfluid motion

$$\begin{aligned} & - \sum_{\mathbf{k}_1 \neq 0} \varepsilon(k_1) \left[\frac{\partial^2 U}{\partial \rho_{\mathbf{k}_1} \partial \rho_{-\mathbf{k}_1}} + \frac{\partial U}{\partial \rho_{\mathbf{k}_1}} \frac{\partial U}{\partial \rho_{-\mathbf{k}_1}} \right] \\ & + \sum_{\mathbf{k}_1 \neq 0} \sum_{\mathbf{k}_2 \neq 0} \frac{\varepsilon(\mathbf{k}_1, \mathbf{k}_2)}{\sqrt{N}} \rho_{\mathbf{k}_1 + \mathbf{k}_2} \left[\frac{\partial U}{\partial \rho_{\mathbf{k}_1} \partial \rho_{\mathbf{k}_2}} + \frac{\partial U}{\partial \rho_{\mathbf{k}_1}} \frac{\partial U}{\partial \rho_{\mathbf{k}_2}} \right] \\ & + \sum_{\mathbf{k}_1 \neq 0} \left[\frac{N}{2V} \nu(k_1) \rho_{\mathbf{k}_1} \rho_{-\mathbf{k}_1} - \frac{1}{4} \rho_{\mathbf{k}_1} \frac{\partial \ln J}{\partial \rho_{\mathbf{k}_1}} \varepsilon(k_1) \right] = E_0 + \sum_{\mathbf{k} \neq 0} \left[\frac{1}{2} \varepsilon(k) + \frac{N}{2V} \nu(k) \right] - \frac{N^2}{2V} \nu(0), \end{aligned} \quad (15)$$

and normal motion

$$\begin{aligned} & - \sum_{\mathbf{k}_1 \neq 0} \varepsilon(k_1) \left[\frac{\partial^2 \varphi}{\partial \rho_{\mathbf{k}_1} \partial \rho_{-\mathbf{k}_1}} + 2 \frac{\partial U}{\partial \rho_{\mathbf{k}_1}} \frac{\partial \varphi}{\partial \rho_{-\mathbf{k}_1}} \right] + \\ & \sum_{\mathbf{k}_1 \neq 0} \sum_{\mathbf{k}_2 \neq 0} \frac{\varepsilon(\mathbf{k}_1, \mathbf{k}_2)}{\sqrt{N}} \rho_{\mathbf{k}_1 + \mathbf{k}_2} \left[\frac{\partial \varphi}{\partial \rho_{\mathbf{k}_1} \partial \rho_{\mathbf{k}_2}} + 2 \frac{\partial U}{\partial \rho_{\mathbf{k}_1}} \frac{\partial \varphi}{\partial \rho_{\mathbf{k}_2}} \right] = E_{ext}(k) \varphi \end{aligned} \quad (16)$$

The wave function ψ must be eigenfunction of a momentum operator [7, 8]:

$$\hat{\mathbf{P}} = - \sum_{\mathbf{k}_1 \neq 0} \hbar \mathbf{k}_1 \rho_{\mathbf{k}_1} \frac{\partial}{\partial \rho_{\mathbf{k}_1}}, \quad \hat{\mathbf{P}} \psi = \mathbf{P} \psi. \quad (17)$$

The wave function $\bar{\psi}$ is eigenfunction of the momentum operator too:

$$J^{1/2} \hat{\mathbf{P}} J^{-1/2} \bar{\psi} = \mathbf{P} \bar{\psi} \implies \hat{\mathbf{P}} \bar{\psi} = \mathbf{P} \bar{\psi}. \quad (18)$$

The proof of this fact is in Appendix A. Then solution of the equations (15) and (16) must obey the condition:

$$\hat{\mathbf{P}} e^U = 0 \cdot e^U, \quad \hat{\mathbf{P}} \varphi = \hbar(\mathbf{k}_1 + \dots + \mathbf{k}_n) \varphi. \quad (19)$$

This means that a center of the system's mass rests and the function φ describes excited states of a system, which are characterized with conserving energy $E_{ext}(k)$ and with conserving momentum $\hbar(\mathbf{k}_1 + \dots + \mathbf{k}_n)$. Thus the function φ describes state with n collective excitations. If an interaction of particles $\nu(k) = 0$ then the wave function ψ is a wave function of free bosons $\psi_{\nu=0} = 1/\sqrt{V^N}$. A proof of this statement is in Appendix B.

In the total case a solution of the equation (15) has a form of correlation series expansion[7, 8]:

$$U = \sum_{\mathbf{k} \neq 0} f(k) \rho_{\mathbf{k}} \rho_{-\mathbf{k}} + \sum_{\mathbf{k} \neq 0} \sum_{\mathbf{q} \neq 0} \frac{g(\mathbf{k}, \mathbf{q})}{\sqrt{N}} \rho_{\mathbf{q}-\mathbf{k}} \rho_{\mathbf{k}} \rho_{-\mathbf{q}} + \dots \quad (20)$$

Analogous expansion for the function φ of state with one CE (as $\hat{\mathbf{P}} \varphi = \hbar \mathbf{k} \varphi$) has a form:

$$\varphi = \rho_{-\mathbf{k}} + \sum_{\mathbf{q} \neq 0} \frac{L(\mathbf{k}, \mathbf{q})}{\sqrt{N}} \rho_{\mathbf{q}-\mathbf{k}} \rho_{-\mathbf{q}} + \dots \quad (21)$$

Unfortunately no small parameter exist in these expansions. And what's more, calculation of higher terms of the expansions is very difficult mathematically. In order to solve this problem we have to use some model considerations and approximations.

We have to make a little remark here. Under *collective excitations* we understand quants of collective motions of macroscopic group of particles, that is with motion of system as a whole. For example: phonon, plasmon, magnon. Under *quasi-particle* we understand a particle interacting with its environment or external field ("dressed" particle). The quasi-particles are characterized by effective mass and they are interacting with effective (screened) potential. The examples of quasi-particle are conduction electron, polaron, Cooper pair.

III. RANDOM PHASE APPROXIMATION

A. The random phases as zeroth approximation

In this section we will shortly formulate zeroth approximation to our problem - RPA or harmonic approximation. The Jacobian and the wave function of a ground state we write in Gauss form:

$$\ln J = \ln C - \frac{1}{2} \sum_{\mathbf{k} \neq 0} \rho_{\mathbf{k}} \rho_{-\mathbf{k}}. \quad (22)$$

$$U = \sum_{\mathbf{k} \neq 0} f(k) \rho_{\mathbf{k}} \rho_{-\mathbf{k}}. \quad (23)$$

Substituting these expressions in the equation (15) and neglecting by powers of $\rho_{\mathbf{k}}$ higher than second power we have:

$$\begin{aligned} \sum_{\mathbf{k} \neq 0} \left[-4\varepsilon(k) f^2(k) + \frac{1}{4} \varepsilon(k) + \frac{N}{2V} \nu(k) \right] \rho_{\mathbf{k}} \rho_{-\mathbf{k}} = \\ E_0 - \frac{N^2}{2V} \nu(0) + \sum_{\mathbf{k} \neq 0} \left[2\varepsilon(k) f(k) + \frac{1}{2} \varepsilon(k) + \frac{N}{2V} \nu(k) \right]. \end{aligned} \quad (24)$$

From this equation we can obtain unknown function f :

$$f(k) = -\frac{1}{4} \sqrt{1 + \frac{2N}{V} \frac{\nu(k)}{\varepsilon(k)}}, \quad (25)$$

and corresponding energy:

$$E_0 = \sum_{\mathbf{k} \neq 0} \frac{1}{2} \sqrt{\varepsilon(k)^2 + \frac{2N}{V} \nu(k) \varepsilon(k)} - \sum_{\mathbf{k} \neq 0} \left[\frac{1}{2} \varepsilon(k) + \frac{N}{2V} \nu(k) \right] + \frac{N^2}{2V} \nu(0). \quad (26)$$

It is necessary to notice that the solutions (25, 26) correspond to the shortened Hamiltonian:

$$\begin{aligned} \hat{H}_{\text{RPA}} = \sum_{\mathbf{k}_1 \neq 0} \varepsilon(k_1) \left(-\frac{\partial^2}{\partial \rho_{\mathbf{k}_1} \partial \rho_{-\mathbf{k}_1}} + \frac{1}{4} \rho_{\mathbf{k}_1} \rho_{-\mathbf{k}_1} - \frac{1}{2} \right) \\ + \frac{N^2}{2V} \nu(0) + \frac{N}{2V} \sum_{\mathbf{k}_1 \neq 0} \nu(k_1) (\rho_{\mathbf{k}_1} \rho_{-\mathbf{k}_1} - 1), \end{aligned} \quad (27)$$

where, unlike the complete Hamiltonian (10), the harmonic terms is kept only.

A structure factor is the most important characteristic of liquids [24]:

$$S(k) = \langle \rho_{\mathbf{k}} \rho_{-\mathbf{k}} \rangle, \quad (28)$$

where the mean value $\langle \rangle$ is calculated with a ground state. This function can be obtained by using the virial theorem:

$$\frac{\delta E}{\delta \nu(k)} = \left\langle \frac{\delta \hat{H}}{\delta \nu(k)} \right\rangle \Rightarrow S(k) = \frac{1}{\sqrt{1 + \frac{2N}{V} \frac{\nu(k)}{\varepsilon(k)}}} = -\frac{1}{4f(k)}. \quad (29)$$

For atoms of helium the potential of interaction is very like to Lennard-Jones potential in the region $r > 2.5A$. In the remaining region the interaction is known badly and various adaption functions have be used here. In order to get over this difficulty the approach with structure factor has been developed in some papers where the potential of interaction is unknown function and the structure factor is taken from experiment data. This means, that the

perturbation theory must be constructed so we must obtain the same structure factor in each approximation. On the present step of calculation the potential of interaction is obtained from (29) in a form:

$$\nu(k) = \frac{V}{2N} \varepsilon(k) \left[\frac{1}{S^2(k)} - 1 \right]. \quad (30)$$

In order to obtain dispersion curve of a CE we assume that $\varphi = \rho_{-\mathbf{k}}$. Substituting its and the function (23) in the equation (16) we have:

$$E_{ext}^{RPA} = \frac{\varepsilon(k)}{S(k)} = \sqrt{\varepsilon(k)^2 + \frac{2N}{V} \nu(k) \varepsilon(k)}. \quad (31)$$

This expression is known as Feynman formula or Bogoliubov specter. The formula (31) is a starting-point in order to obtain the dispersion curve of a CE which is consistent with the experimental specter.

B. Formulation of the oscillator model

The Hamiltonian (27) is sum of separate terms where each of them is characterized by own wave vector \mathbf{k} . This fact means that infinity number of independent motions can be in a system, and every of them is characterized by wave vector \mathbf{k} and some energy $\xi(k)$. For the each independent motion Schrodinger equation is written as follows:

$$-\frac{\hbar^2}{2(m/k^2)} \frac{\partial^2 \bar{\psi}}{\partial \rho_{\mathbf{k}} \partial \rho_{-\mathbf{k}}} + \left[\frac{1}{4} \varepsilon(k) + \frac{N}{2V} \nu(k) \right] \rho_{\mathbf{k}} \rho_{-\mathbf{k}} \bar{\psi} = \xi(k) \bar{\psi}. \quad (32)$$

This equation is similar to an equation of a harmonic oscillator, where

$$\frac{\hbar^2 k^2}{2m} \equiv \frac{\hbar^2}{2M} \Rightarrow M \equiv \frac{m}{k^2}, \quad \frac{1}{4} \varepsilon(k) + \frac{N}{2V} \nu(k) \equiv \frac{1}{2} M \omega^2. \quad (33)$$

$\xi(k)$ is energy of a oscillator with the wave vector \mathbf{k} :

$$\xi(k) = \hbar \omega(k) \left(\frac{1}{2} + n \right) = \frac{\varepsilon(k)}{S(k)} \left(\frac{1}{2} + n \right), \quad (34)$$

and the ground state energy of a system is

$$E_0 = \sum_{\mathbf{k} \neq 0} \frac{1}{2} \hbar \omega(k) - \sum_{\mathbf{k} \neq 0} \left[\frac{1}{2} \varepsilon(k) + \frac{N}{2V} \nu(k) \right] + \frac{N^2}{2V} \nu(0). \quad (35)$$

The wave function of a ground state has the form:

$$\bar{\psi} \sim \exp \left(-\frac{\rho_{\mathbf{k}} \rho_{-\mathbf{k}}}{2l^2} \right) = e^{f(k) \rho_{\mathbf{k}} \rho_{-\mathbf{k}}}, \quad l^2 = \frac{\hbar}{M \omega} \quad (36)$$

where l is a oscillator length. The structure factor can be found from the virial theorem for an oscillator:

$$\frac{M \omega^2}{2} \langle \rho_{\mathbf{k}} \rho_{-\mathbf{k}} \rangle = \frac{\xi(k)}{2} \Rightarrow S = \frac{\varepsilon(k)}{\hbar \omega(k)} [1 + 2 \langle n \rangle], \quad (37)$$

and the state $n = 0$ corresponds to a ground state. A structure factor $S(k)$ is dispersion of a oscillator with a wave vector \mathbf{k} .

On the assumption of the aforesaid we can formulate the oscillator model of Bose liquid. A quantum liquid represents a totality of harmonic oscillators. Each oscillator is a vibration mode of density of liquid. A ground state energy of a system is a sum of energies of ground states of these oscillators. Availability in a system n collective excitations corresponds to a state with n excited oscillators. The oscillator model is a model of continuum medium and it is incorrect on distances between neighboring atoms. The calculation of short-range correlations is very difficult because all anharmonic terms in the Hamiltonian (10) and in the Jacobian (12) must be calculated. Moreover if we shall try to calculate them with help ordinary perturbation theory then we shall have divergent integrals. This difficulty appears due contribution of above-mentioned superfluous degrees of freedom. In the next sections we shall formulate the method of calculation of the anharmonic terms remaining in limits of the oscillator model.

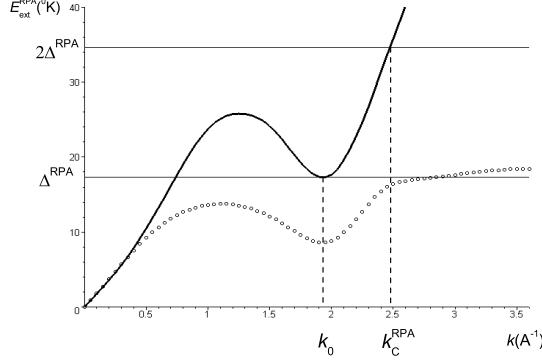


Figure 1: The spectrum of CE calculated with the Feynman formula (solid line) and measured in experiment (dotted line). Decay of excitation on two rotons is possible in the region $k > k_C^{RPA}$ where $E_{ext}^{RPA}(k) > 2\Delta^{RPA}$. The experimental curve has a "shelf" at $k > 2.5\text{A}^{-1}$ where decay of CE into two rotons is observed.

IV. THE SPECTRUM OF COLLECTIVE EXCITATION

A. Decay of collective excitation

The wave function of a excited state $\varphi = \rho_{-\mathbf{k}}$ leads to the well known Feynman formula (31). We can see on Fig.1 that the dispersion curve in RPA coincides with the experimental dispersion curve in the limit $k \rightarrow 0$ only. Its basic contrast to the real spectrum is that that Feynman formula doesn't take into account decay of CE and the end of a dispersion curve accordingly.

Let's consider the wave function of CE (21) with momentum $\hbar\mathbf{k}$ where $L(\mathbf{k}, \mathbf{q})$ is the unknown function. In our interpretation the second term in (21) describes a process of decay of CE with momentum $\hbar\mathbf{k}$ and energy $E_{ext}(k)$ into two excitations with momentums $\hbar(\mathbf{k} - \mathbf{q})$ and $\hbar\mathbf{q}$ and with energies $E_{ext}(|\mathbf{k} - \mathbf{q}|)$ and $E_{ext}(q)$ accordingly. Decay of CE can occur if energy is conserved:

$$E_{ext}(k) = E_{ext}(|\mathbf{k} - \mathbf{q}|) + E_{ext}(q). \quad (38)$$

Such situation corresponds to the case (A) on Fig.2. This process causes damping of a CE and the end of the dispersion curve [26]. In those parts of the energy spectrum where the equality (38) isn't true, CE is stable. In this case the term in (21) with the multiplier $\rho_{\mathbf{q}-\mathbf{k}}\rho_{-\mathbf{q}}$ describes processes of virtual decay and creation of CE. This causes trivial renormalization of the energy spectrum of CE. This situation corresponds to the case (B) on Fig.2. Higher approximations in (21) describe decays of CE on three and more excitations. We suppose that such processes is improbable and we shall not consider them in this paper.

Substituting the wave function (21) in the equation (16) we can obtain the function

$$L(\mathbf{k}, \mathbf{q}) = \frac{\varepsilon(\mathbf{k}, \mathbf{q})S^{-1}(q)}{E_{ext}(k) - E_{ext}^{RPA}(|\mathbf{k} - \mathbf{q}|) - E_{ext}^{RPA}(q)}, \quad (39)$$

where $E_{ext}^{RPA}(q)$ is spectrum of CE in RPA - the formula (31). And besides we can obtain the dispersion law of CE in a form:

$$\begin{aligned} E_{ext}(k) &= E_{ext}^{RPA}(k) + \frac{2}{N} \sum_{\mathbf{q} \neq 0} L\varepsilon(\mathbf{k} - \mathbf{q}, \mathbf{q}) \\ &= E_{ext}^{RPA}(k) - \frac{2}{N} \sum_{\mathbf{q} \neq 0} \frac{\varepsilon(\mathbf{k}, \mathbf{q})\varepsilon(\mathbf{k} - \mathbf{q}, \mathbf{q})S^{-1}(q)}{E_{ext}^{RPA}(|\mathbf{k} - \mathbf{q}|) + E_{ext}^{RPA}(q) - E_{ext}(k)}. \end{aligned} \quad (40)$$

This equation is an integral equation for unknown function $E_{ext}(k)$. The first term corresponds to the term $\rho_{-\mathbf{k}}$ in the wave function (21) and it is Bogoliubov-Feynman spectrum. The second term has a singularity in those points q where the equality is executed:

$$E_{ext}(k) = E_{ext}^{RPA}(|\mathbf{k} - \mathbf{q}|) + E_{ext}^{RPA}(q). \quad (41)$$

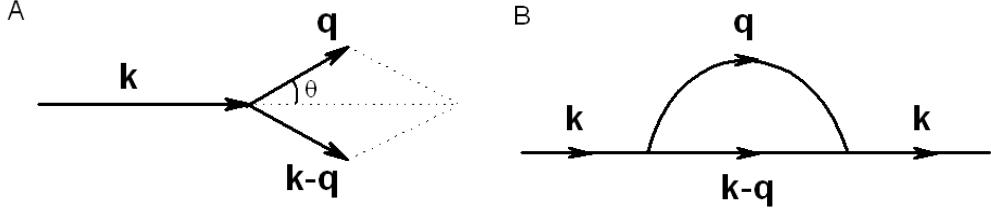


Figure 2: The processes described by the second term in the wave function (21). The process (A) is a real decay, the process (B) is a virtual decay of CE with momentum $\hbar\mathbf{k}$ into two excitations with momentums $\hbar(\mathbf{k} - \mathbf{q})$ and $\hbar\mathbf{q}$.

This equation likes the expression (38) and it is condition of decay of CE. Inexactitude of this condition is in that that a real CE $E_{ext}(k)$ decays on excitations with a spectrum calculated in RPA: $E_{ext}^{RPA}(q)$. However in the region of large q the theoretical dispersion law of CE differs from the real law $E_{ext}(q)$ essentially. We shall get over this difficulty later. For the present we shall work with the condition (41).

CE is stable at those k where the equation (41) hasn't solutions. Damping of CE appears in a threshold of decay $k = k_C$. At the point $k = k_C$ a solution of the equation (41) appears at first, and the right sides of the equations (38) and (41) have extremum [26] as a function of q . We can see on Fig.1 that the extremum for the function $E_{ext}^{RPA}(q)$ is the roton minimum. In its neighborhood the spectrum of collective excitations has a view:

$$E_{ext}^{RPA}(q) = \Delta^{RPA} + \frac{\hbar^2}{2\mu}(q - k_0)^2, \quad (42)$$

$$E_{ext}^{RPA}(|\mathbf{k} - \mathbf{q}|) = \Delta^{RPA} + \frac{\hbar^2}{2\mu}(|\mathbf{k} - \mathbf{q}| - k_0)^2, \quad (43)$$

where $\Delta^{RPA} = 17.3^0 K$ is the roton gap in RPA (in degrees of Kelvin), $\mu = 0.09m$ is effective mass of a roton in RPA, k_0 is momentum corresponding to the roton gap (in angstroms). We can suppose that $k_0 = k_0^{RPA} = 1.93 A^{-1}$ in all approximations. The condition (41) is executed when $E_{ext}(k) = 2\Delta^{RPA}$. Then the underintegral function in the spectrum (40) at $k = k_C$ has a singularity in the point $q = q_0$. This means decay of CE with momentum $\hbar k_C$ into two rotons with momentums $\hbar k_0$ each. Hence we have the connection $k_C = 2k_0 \cos \theta_c$ where $2\theta_c$ is a angle of recession of two rotons.

Let's evaluate the integral in (40) in the point $k = k_C$. Let's consider asymptotics of the underintegral function at $q \rightarrow \infty$ where $S(q) \rightarrow 1$. If a vector \mathbf{k} is directed along the axe Oz then $\mathbf{k}\mathbf{q} = kq \cos \theta$. Let's use the expressions (42) and (43) in the denominator of (40) as they determine the singularity at $q = k_0$, and we have $|\mathbf{k} - \mathbf{q}| \approx q$. Then the integral in (40) is written in a form:

$$\begin{aligned} & \frac{2}{N} \frac{V}{(2\pi)^3} \left(\frac{\hbar^2}{2m} \right)^2 \left(\int \frac{(qk \cos \theta)^2 q^2 dq \sin \theta d\theta d\varphi}{2\Delta^{RPA} - E_{ext} + \frac{\hbar^2}{\mu}(q - k_0)^2} \right. \\ & \left. - \int \frac{qk \cos \theta q^2 dq \sin \theta d\theta d\varphi}{2\Delta^{RPA} - E_{ext} + \frac{\hbar^2}{\mu}(q - k_0)^2} \right) \end{aligned} \quad (44)$$

$$= \frac{2}{N} \frac{V}{(2\pi)^2} \left(\frac{\hbar^2}{2m} \right)^2 \frac{2}{3} k^2 \int \frac{q^4 dq}{2\Delta^{RPA} - E_{ext} + \frac{\hbar^2}{\mu}(q - k_0)^2}. \quad (45)$$

In the second integral in (44) integration by θ gives zero. Taking into consideration $E_{ext}(k_C) = 2\Delta^{RPA}$ we can rewrite (45) as

$$\frac{2}{N} \frac{V}{(2\pi)^2} \left(\frac{\hbar^2}{2m} \right)^2 \frac{2}{3} k^2 \frac{\mu}{\hbar^2} \int_0^\infty \frac{q^4 dq}{(q - k_0)^2} \rightarrow \infty \quad (46)$$

The underintegral function has a pole in the point k_0 that results in some complex addendum in an energy of CE: $E_{ext} - i/\tau$. This means damping and limited life time $\sim \tau$ of CE. Moreover we can see that *ultraviolet divergence* takes

place. Hence the second term in the formula (40) for the spectrum of CE aspires to infinity: $E_{ext}(k) = E_{ext}^{RPA}(k) + \infty$. The nature of this divergence lies in the following.

The constraint equation (11) removes the superfluous degrees of freedom appeared at nonunitarian transition to collective coordinates. These degrees of freedom are connected with continuum representation of primarily discreet medium. Each superfluous degree of freedom gives contribution to energy. Neglect of all anharmonic terms in the Jacobian (12) and in the Hamiltonian (10) doesn't give possibility to delete this nonphysical energy. Contribution of the superfluous degrees of freedom results in infinity in the integral (40).

The method of removal of the divergence is cut-off of the integral (40) to some unknown value of a wave vector $q = q_m$. By analogy with [15, 26] let's introduce new variables q'_z q'_ρ in accordance with the definition:

$$\begin{aligned} q_x &= (k_0 \sin \theta_c + q'_\rho) \cos \varphi & q_y &= (k_0 \sin \theta_c + q'_\rho) \sin \varphi \\ q_z &= k_0 \cos \theta_c + q'_z & d^3 q &= q^2 dq \sin \theta d\theta d\varphi = (k_0 \sin \theta_c + q'_\rho) dq'_\rho dq'_z d\varphi. \end{aligned} \quad (47)$$

In a neighborhood of the threshold of decay $k = k_C$ (where $E_{ext} \rightarrow 2\Delta^{RPA}$) the underintegral function in (40) has a pole in the point $[q = k_0, \theta = \theta_c]$. We have $|q'_z| \ll k_0$ and $|q'_\rho| \ll k_0$ in a neighborhood of the singularity $q \rightarrow k_0, \theta \rightarrow \theta_c$. Then we can write expansions near the singularity:

$$\begin{aligned} d^3 q &\approx k_0 \sin \theta_c dq'_\rho dq'_z d\varphi, & q &\approx k_0 + q'_z \cos \theta_c + q'_\rho \sin \theta_c \\ |\mathbf{k} - \mathbf{q}| &\approx k_0 + q'_\rho \sin \theta_c - q'_z \cos \theta_c, & \mathbf{k}\mathbf{q} &\approx k k_0 \cos \theta_c + O(q'_\rho, q'_z) \\ (\mathbf{k} - \mathbf{q})\mathbf{q} &\approx k_0^2 \cos 2\theta_c + O(q'_\rho, q'_z). \end{aligned} \quad (48)$$

In the limit $q \rightarrow k_0$ energies of products of decay have a view:

$$\begin{aligned} E_{ext}^{RPA}(q) &= \Delta^{RPA} + \frac{\hbar^2}{2\mu} (q'_z \cos \theta_c + q'_\rho \sin \theta_c)^2, \\ E_{ext}^{RPA}(|\mathbf{k} - \mathbf{q}|) &= \Delta^{RPA} + \frac{\hbar^2}{2\mu} (q'_\rho \sin \theta_c - q'_z \cos \theta_c)^2. \end{aligned} \quad (49)$$

Let's consider the spectrum of CE (40) in some neighborhood of the threshold of decay $k \rightarrow k_C$. Decomposing the underintegral function near the pole $q \rightarrow k_0, \theta \rightarrow \theta_c$ using the expansions (48) and (49) we have::

$$-\frac{2}{N} \sum_{\mathbf{q} \neq 0} \frac{\varepsilon(\mathbf{k}, \mathbf{q}) \varepsilon(\mathbf{k} - \mathbf{q}, \mathbf{q}) S^{-1}(q)}{E_{ext}^{RPA}(|\mathbf{k} - \mathbf{q}|) + E_{ext}^{RPA}(q) - E_{ext}(k)} \quad (50)$$

$$\begin{aligned} &\rightarrow -\frac{2}{N} \frac{V}{(2\pi)^2} \left(\frac{\hbar^2 k_0^2}{2m} \right)^2 \frac{\mu}{\hbar^2} S^{-1}(k_0) \cos 2\theta_c \cos \theta_c \sin \theta_c \\ &\times k \int \frac{dq'_z dq'_\rho}{\frac{\mu}{\hbar^2} (2\Delta^{RPA} - E_{ext}) + [(q'_\rho)^2 \sin^2 \theta_c + (q'_z)^2 \cos^2 \theta_c]}. \end{aligned} \quad (51)$$

However the integrals (50) and (51) are not equal because transition to the limit $q \rightarrow k_0, \theta \rightarrow \theta_c$ has been done in the underintegral function. Integration by all q -space results to infinity. Then we must introduce some cut-off parameter ρ so as the integral (51) is finite on the one hand, and the integrals (50) and (51) are approximately equal in a neighborhood of the point $q = k_0$: $k_0 - \rho < q < k_0 + \rho$ on the other hand.

Now let's introduce the polar coordinates:

$$\begin{aligned} q'_\rho \sin \theta_c &= \rho \cos \psi \\ q'_z \cos \theta_c &= \rho \sin \psi \\ dq'_\rho dq'_z &= \frac{\rho d\rho d\psi}{\cos \theta_c \sin \theta_c}. \end{aligned} \quad (52)$$

Then the integral (51) is reduced to the form:

$$\begin{aligned} &-\frac{2}{N} \frac{V}{2\pi} \varepsilon^2(k_0) \frac{\mu \cos 2\theta_c}{\hbar^2} S^{-1}(k_0) k \int_0^\rho \frac{\rho d\rho}{\frac{\mu}{\hbar^2} (2\Delta^{RPA} - E_{ext}) + \rho^2} \\ &= \frac{V}{N} \frac{\varepsilon^2(k_0) \mu \cos 2\theta_c}{2\pi \hbar^2 S(k_0)} k \ln \left[\frac{(2\Delta^{RPA} - E_{ext}) \mu / \hbar^2}{(2\Delta^{RPA} - E_{ext}) \mu / \hbar^2 + \rho^2} \right], \end{aligned} \quad (53)$$

We can see that the final result depends on the unknown parameter ρ which determines a integration domain near the point $q = k_0$. Then ρ must satisfy the requirements:

1. The point k_C is the threshold of decay: $E_{ext}(k_C) = 2\Delta^{RPA}$
2. On Fig.1 we can see that Feynman formula $E_{ext}^{RPA}(k) = \varepsilon(k)/S(k)$ in the limit $k \rightarrow 0$ gives a sound mode $c\hbar k$ with a correct sound velocity c . Hence *the total spectrum (40) must have the same asymptotic*:

$$E_{ext}(k \rightarrow 0) = E_{ext}^{RPA}(k \rightarrow 0) = c\hbar k. \quad (54)$$

The second term of the expression (40) in the limit $k \rightarrow k_C$ is the integral (51). We must join these two asymptotics in a single expression which is a sought dispersion curve. Hence the parameter ρ is function of variable k : $\rho = \rho(k)$.

3. The parameter ρ must ensure approximate equality of the integrals (50) and (51) in a neighborhood of the point $q = k_0$. We have decomposed the underintegral function in (50) near the point $q = k_0$ in power series of ρ/k_0 :

$$\begin{aligned} q &\approx k_0 + q'_z \cos \theta_c + q'_\rho \sin \theta_c \approx k_0 (1 + \rho/k_0), \\ |\mathbf{k} - \mathbf{q}| &\approx k_0 + q'_\rho \sin \theta_c - q'_z \cos \theta_c \approx k_0 (1 + \rho/k_0), \\ k_0 \sin \theta_c + q'_\rho &\approx k_0 \sin \theta_c \left(1 + \frac{\rho}{k_0 \sin \theta_c}\right) \Rightarrow \frac{\rho}{k_0 \sin \theta_c} \ll 1. \end{aligned} \quad (55)$$

Hence $\rho(k)$ must satisfy the inequality (55).

4. The spectrum of CE (40) must be determined with the parameter $\rho(k)$ by self-consistent way.

In order to meet the requirements (1-4) we must assume that

$$\frac{(2\Delta^{RPA} - E_{ext})\mu/\hbar^2}{(2\Delta^{RPA} - E_{ext})\mu/\hbar^2 + \rho^2} = \frac{2\chi\Delta^{RPA} - E_{ext}}{2\chi\Delta^{RPA}}, \quad (56)$$

then

$$\rho^2(k) = \frac{\mu}{\hbar^2} \frac{E_{ext}(k) [2\Delta^{RPA} - E_{ext}(k)]}{2\chi\Delta^{RPA} - E_{ext}(k)}, \quad (57)$$

where the parameter χ is determined by the condition (1) $E_{ext}(k_C) = 2\Delta^{RPA}$:

$$\chi = \left[1 - \exp \left(\frac{2\Delta^{RPA} - E_{ext}^{RPA}(k_C)}{\alpha k_C} \right) \right]^{-1}. \quad (58)$$

As a result the dispersion equation for CE (40) has a form:

$$E_{ext}(k) = E_{ext}^{RPA}(k) + \alpha k \ln \left[\frac{2\chi\Delta^{RPA} - E_{ext}(k)}{2\chi\Delta^{RPA}} \right], \quad (59)$$

where

$$\alpha = \frac{V}{N} \frac{\varepsilon^2(k_0)\mu \cos 2\theta_c}{2\pi\hbar^2 S(k_0)}, \quad \cos \theta_c = \frac{k_C}{2k_0}. \quad (60)$$

We can see that *the parameter ρ is determined by the spectrum $E_{ext}(k)$. On the other hand, ρ determines this spectrum. $E_{ext}(k)$ doesn't contain ρ in an explicit form. Hence $E_{ext}(k)$ is determined with the parameter ρ by self-consistent way.*

In the long-wave asymptotic $E_{ext}^{RPA}(k \rightarrow 0) = c\hbar k + Ak^3$, $A = \text{const} > 0$. Then it is easily to show that

$$E_{ext}(k \rightarrow 0) = c\hbar k - \frac{c\hbar\alpha}{2\chi\Delta^{RPA}} k^2 + Ak^3 - \dots \quad (61)$$

This means that the spectrum E_{ext} is joined with Feynman formula at small k and describes the sound mode. Thus the condition (2) is satisfied. However we can see that even powers of k exist in the expansion (61). But this

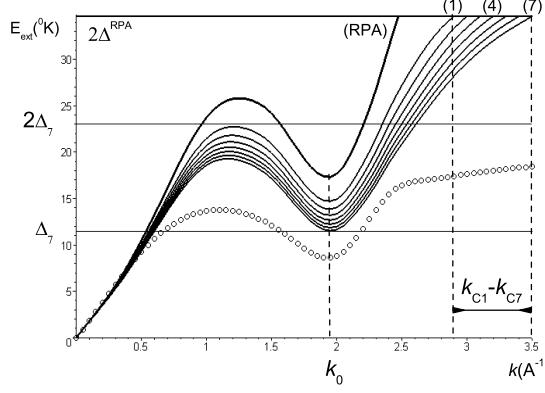


Figure 3: The spectrums of collective excitations. The line (RPA) is spectrum corresponding to Feynman formula (31). The lines **1-7** are dispersion curves obtained as numerical solution of the equation (59) with parameters k_C equaling $2.9A^{-1}$, $3.0A^{-1}$, $3.1A^{-1}$, $3.2A^{-1}$, $3.3A^{-1}$, $3.4A^{-1}$, $3.5A^{-1}$ accordingly. The dotted line is experimental spectrum of CE [28] at low temperature. The dispersion curves **1-7** finish in the points $k = k_C$ on the line $2\Delta^{RPA}$.

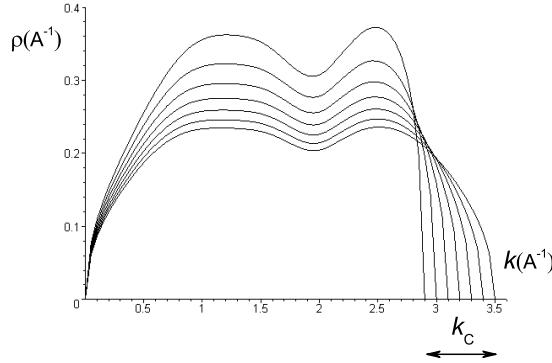


Figure 4: The parameters of cutt-off $\rho(k)$ for the integral (53). The curves $\rho(k)$ correspond to the curves **1-7** on Fig.(3). A domain of the functions $\rho(k)$ is $0 \leq k \leq k_C$.

doesn't contradict to isotropy of liquid because the dispersion equation (59) depends on modulus of wave vector only $E_{ext} = E_{ext}(|\mathbf{k}|)$.

We can see that the dispersion curve (59) depends on the parameter k_C (threshold of decay) strongly. This is *free parameter* being in the limits $k_C^{RPA} \leq k_C \leq 2k_0$, therefore we have a family of curves shown in Fig.3. Corresponding cut-off parameters ρ which joins the long-wave asymptotic $k \rightarrow 0$ with the asymptotic $k \rightarrow k_C$ is shown on Fig.4.

A domain of the function $\rho(k)$ is $0 \leq k \leq k_C$ because at $k > k_C$ we have $\rho(k)^2 < 0$. This means that the dispersion curve (59) can not be continued to the region $k > k_C$ (though the function $E_{ext}(k)$ exists formally). Hence *the point k_C is end point of the spectrum*. For the typical value $k_C = 3.2A^{-1}$ we have: $\max(\rho/k_0) = 0.28/1.93 = 0.15 \ll 1$ and $\max(\rho/k_0/\sin \theta_c) = 0.27 < 1$. This means that the condition (3) is satisfied.

The dispersion curves **1-7** on Fig.3 are more close to the experimental curve than $E_{ext}^{RPA}(k)$, however they have an essential defect. CE with the dispersion curve $E_{ext}(k)$ decays into excitations with spectrum calculated in RPA: $E_{ext}^{RPA}(q)$ and $E_{ext}^{RPA}(|\mathbf{k} - \mathbf{q}|)$. This situation corresponds to conservation of energy in the form (41). In reality the law of conservation of energy is the form (38) where energy of decaying CE and energies of decay products are determined with the *real* spectrum. We can see on Fig.3 that the dispersion curve $E_{ext}(k)$ differs from the curve $E_{ext}^{RPA}(k)$ essentially in the region of big k . So, the new value of a roton gap is $\Delta < \Delta^{RPA}$. This means that energy is not conserved at decay of CE into two rotons. In other words the dispersion curve $E_{ext}(k)$ is not self-consistent.

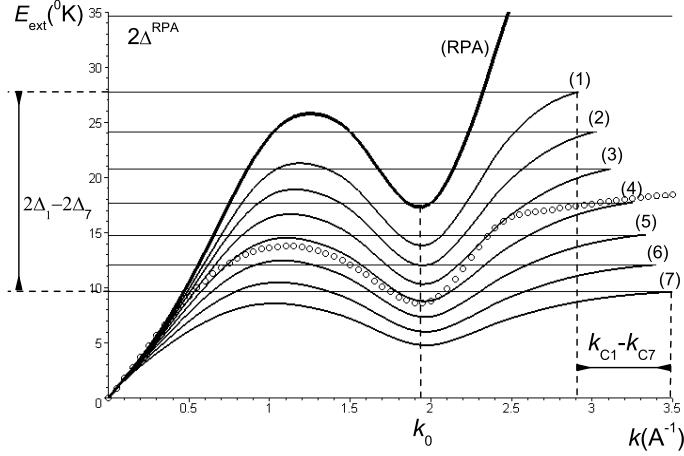


Figure 5: The spectrums of collective excitations. The line (RPA) is spectrum corresponding to Feynman formula (31). The lines 1-7 are dispersion curves obtained by numerical solution of the set of equations (62) and (64) with parameters $k_C = 2.9A^{-1}$, $3.0A^{-1}$, $3.1A^{-1}$, $3.2A^{-1}$, $3.3A^{-1}$, $3.4A^{-1}$, $3.5A^{-1}$ accordingly. The dotted line is the experimental spectrum of CE [28]. The dispersion curves 1-7 are ended in the points $k = k_C$ on the lines $2\Delta_{1-7}$.

B. The self-consistent form of the dispersion curve $E_{ext}(k)$

The dispersion equation $E_{ext}(k)$ is self-consistent if *an energy of the end point of the spectrum is equal to the doubled roton gap of this spectrum*: $E_{ext}(k_C) = 2\Delta$. Let the new spectrum has a form

$$E_{ext}(k) = E_{ext}^{RPA}(k) + \alpha k \ln \left[\frac{2\tilde{\Delta} - E_{ext}(k)}{2\tilde{\Delta}} \right], \quad (62)$$

where the parameter $\tilde{\Delta}$ must be such that the condition of self-consistency is executed:

$$\left\{ \begin{array}{l} E_{ext}(k_C) = 2\Delta \\ E_{ext}(k_0) = \Delta \end{array} \right\}. \quad (63)$$

The parameter Δ is the *new* roton gap in the point k_0 . The the set of equations (63) in a expanded form is

$$\left\{ \begin{array}{l} 2\Delta = E_{ext}^{RPA}(k_C) + \alpha k_C \ln \left(1 + \frac{\Delta}{2\tilde{\Delta}} \right) \\ \Delta = E_{ext}^{RPA}(k_0) + \alpha k_0 \ln \left(1 + \frac{\Delta}{2\tilde{\Delta}} \right) \end{array} \right\}. \quad (64)$$

The set of equations (62) and (64) determines a dispersion curve of CE. The results of numerical solution of these equations with $k_C = 2.9A^{-1} - 3.5A^{-1}$ are shown on Fig.5. The curves 1-7 on Fig.5 correspond to the curves 1-7 on Fig.4. As in the previous case the spectrum depends on its end point k_C strongly though sound velocity $\hbar c = \lim_{k \rightarrow 0} \frac{\partial E_{ext}}{\partial k}$ is the same in the all cases. We can see a weak pinning of the curve $E_{ext}(k)$ to the line 2Δ . This means hybridization of the dispersion curve of CE with a two-roton level [27]. On the rest of regions an energy $E_{ext}(k)$ is determined by a state with one CE (phonon or roton) completely. We have the family of curves E_{ext} and we cannot obtain the point k_C on this step of calculations. In order to do this we must generalize the oscillator model formulated in the Subsection III B.

V. THE GROUND STATE AND BOSE CONDENSATE

A. Formalism of effective mass in the oscillator model.

In this section our problem is calculation of a ground state energy taking into consideration the anharmonic correction in the equation (15). If we calculate this corrections then we have a divergence $E_0 = E_0^{RPA} + \infty$ as in the

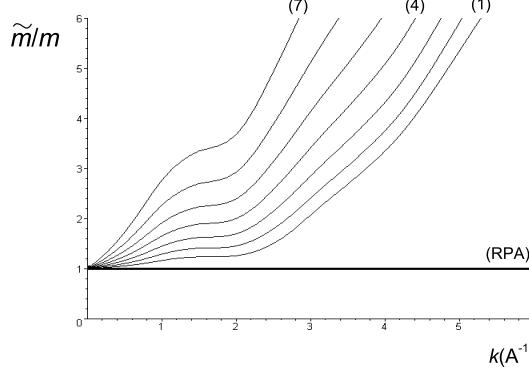


Figure 6: The effective masses of quasi-particles in units of mass of ${}^4\text{He}$: \tilde{m}/m . The curve (RPA) corresponds to a "naked" particle. The numbers 1-7 correspond to curves in Fig.5. The functions $\tilde{m}(k)$ have a sense in the range $k < k_C$ only.

Section IV. In order to overcome this difficulty let's use the oscillator model formulated in the Section III where a superfluid liquid is a totality of independent harmonic oscillators with frequencies $E_{ext}^{RPA}(k) = \varepsilon(k)/S(k)$. In higher approximations we have another spectrums of CE $E_{ext}(k)$: the set of equations (62) and (64). However this spectrum can be written by analogy with RPA as

$$E_{ext}(k) = \frac{\tilde{\varepsilon}(k)}{S(k)}, \quad \text{where} \quad \tilde{\varepsilon}(k) = \frac{\hbar^2 k^2}{2\tilde{m}(k)}. \quad (65)$$

Due to contribution of anharmonic terms of the Hamiltonian (10) mass of a *particle* m is renormalized to mass of a *quasi-particle* $\tilde{m}(k)$. Continuing the analogy with RPA we can write the effective Hamiltonian having a form as in RPA:

$$\begin{aligned} \hat{H}_{\text{eff}} = & \sum_{\mathbf{k}_1 \neq 0} \tilde{\varepsilon}(k_1) \left(-\frac{\partial^2}{\partial \rho_{\mathbf{k}_1} \partial \rho_{-\mathbf{k}_1}} + \frac{1}{4} \rho_{\mathbf{k}_1} \rho_{-\mathbf{k}_1} - \frac{1}{2} \right) \\ & + \frac{N^2}{2V} \tilde{\nu}(0) + \frac{N}{2V} \sum_{\mathbf{k}_1 \neq 0} \tilde{\nu}(k_1) \left(\rho_{\mathbf{k}_1} \rho_{-\mathbf{k}_1} - 1 \right) \end{aligned} \quad (66)$$

and the corresponding Jacobian:

$$\ln J = \ln \left(V^N \prod'_{\mathbf{k} \neq 0} \frac{1}{\pi} \right) - \frac{1}{2} \sum_{\mathbf{k} \neq 0} \rho_{\mathbf{k}} \rho_{-\mathbf{k}}, \quad (67)$$

where the normalization requirement (5) has been used.

We must obtain the same structure factor $S(k)$ in any approximation because this function is given by experiment. Therefore we must introduce the effective interaction $\tilde{\nu}(k)$ as follows:

$$\begin{aligned} S(k) = -\frac{1}{4f(k)} &= \frac{1}{\sqrt{1 + \frac{2N}{V} \frac{\nu(k)}{\varepsilon(k)}}} = \frac{1}{\sqrt{1 + \frac{2N}{V} \frac{\tilde{\nu}(k)}{\tilde{\varepsilon}(k)}}} \\ \Rightarrow \tilde{\nu}(k) &= \tilde{\varepsilon}(k) \frac{V}{2N} \left[\frac{1}{S(k)^2} - 1 \right]. \end{aligned} \quad (68)$$

The effective masses and interactions are shown in Fig.6 and Fig.7. It is necessary to notice that $\tilde{m}(k \rightarrow 0) = m$ and $\tilde{\nu}(0) = \nu(0)$.

As a result we have the totality of harmonic oscillators again. These oscillators have the same dispersions $\langle \rho_{\mathbf{k}} \rho_{-\mathbf{k}} \rangle$ as in RPA. However they have another frequencies $E_{ext}(k)$ instead of the old $E_{ext}^{RPA}(k)$. Difference between new

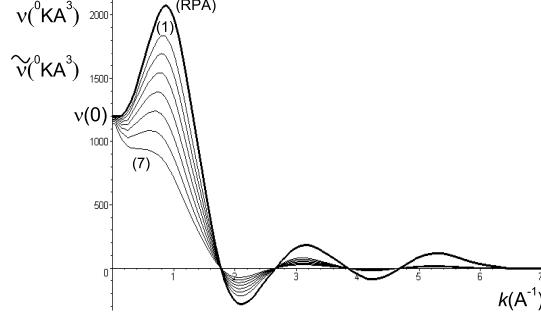


Figure 7: The potential of interaction between particles (RPA) and between quasi-particles (1-7).

frequencies and the initial frequencies depends on the aforesaid anharmonicities. A motion equation for each oscillator can be written as the equation (32) but with an effective mass and an effective interaction:

$$-\frac{\hbar^2}{2\tilde{M}(k)} \frac{\partial^2 \bar{\psi}}{\partial \rho_{\mathbf{k}} \partial \rho_{-\mathbf{k}}} + \left[\frac{1}{4} \tilde{\varepsilon}(k) + \frac{N}{2V} \tilde{\nu}(k) \right] \rho_{\mathbf{k}} \rho_{-\mathbf{k}} \bar{\psi} = \tilde{\xi}(k) \bar{\psi}. \quad (69)$$

A ground state energy can be rewritten as follows:

$$E_0 = \sum_{\mathbf{k} \neq 0} \frac{1}{2} \frac{\tilde{\varepsilon}(k)}{S(k)} - \sum_{\mathbf{k} \neq 0} \left[\frac{1}{2} \tilde{\varepsilon}(k) + \frac{N}{2V} \tilde{\nu}(k) \right] + \frac{N^2}{2V} \tilde{\nu}(0). \quad (70)$$

It is necessary to notice that we must do transition from summation on \mathbf{k} to integration as follows:

$$\sum_{\mathbf{k} \neq 0} \rightarrow \frac{V}{(2\pi)^3} \int_0^{k_C} k^2 dk. \quad (71)$$

The results of calculation of a ground state energy (per one atom) for particles' system (RPA) and for quasi-particles' system (the curves 1-7) are:

$$\begin{aligned} E_0^{RPA}/N &= -13.78^\circ K, & E_0^{(1)}/N &= -8.73^\circ K, & E_0^{(2)}/N &= -6.66^\circ K \\ E_0^{(3)}/N &= -4.62^\circ K & E_0^{(4)}/N &= -2.60^\circ K & E_0^{(5)}/N &= -0.59^\circ K \\ E_0^{(6)}/N &= 1.43^\circ K & E_0^{(7)}/N &= 3.43^\circ K \end{aligned} \quad (72)$$

We can see that existence of the end point of a dispersion curve causes some increase of a ground state energy. Since k_C is unknown then we cannot select a value of this energy. The energy E_0 is minimal at $k_C < 2.9 A^{-1}$. But the important condition for oscillator frequencies $\hbar\omega = E_{ext}$ exists yet.

If velocity of a CE $\frac{\partial E_{ext}}{\partial p} \equiv \frac{\partial \hbar\omega}{\partial p}$ is more than a sound velocity in a system then CE radiates a phonon [26]. The regions of a dispersion curve, where such is possible, is unstable. We have formulated the oscillator model of Bose liquid where its ground state energy is a sum of energies of ground states of oscillators representing the liquid. In other words the ground state is an infinity number of *virtual* collective excitations. Then *in order to determine a ground state as a state with minimal energy a dispersion curve $E_{ext}(k)$ must be stable*. It means that for all k the inequality must be executed:

$$\frac{\partial E_{ext}(k)}{\partial k} \equiv < \lim_{k \rightarrow 0} \frac{\partial E_{ext}(k)}{\partial k} = c\hbar, \quad (73)$$

If this condition is not satisfied then the ground state falls apart. The plots of CE's velocity is shown in Fig.8. The curves 4-7 satisfy the criterium (73). Hence possible end points present in the interval $k_C = 3.2 A^{-1} \div 2k_0$ only. The spectrum 4 corresponds to the minimal value $k_C = 3.2 A^{-1}$. On the contrary, the curve (RPA) is unstable almost completely.

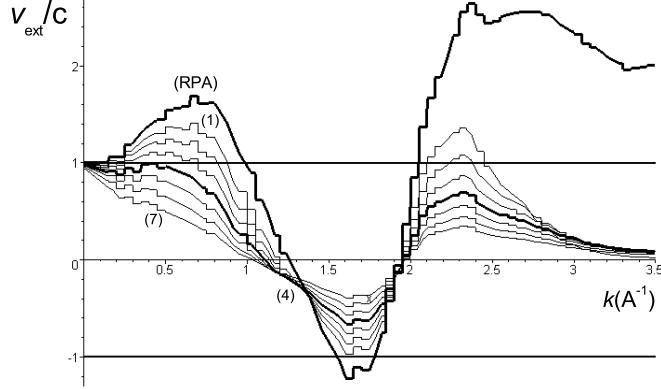


Figure 8: The velocities of collective excitations $v_{ext}(k) = \frac{\partial E_{ext}(k)}{\hbar \partial k}$ per units of a sound velocity c . $v_{ext}(k) < 0$ means that the velocity and the momentum \mathbf{k} have contrary directions. The curve (RPA) corresponds to Feynman spectrum. The curves 1-7 correspond to the curves on Fig.5. The condition (73) is satisfied for the curves where $|v_{ext}(k)|/c < 1$ for all k . The spectrums 4-7 are stable about radiation of a phonon. The spectrum 4 (with $k_C = 3.2A^{-1}$, it is marked by bold line) is on the limit of stability.

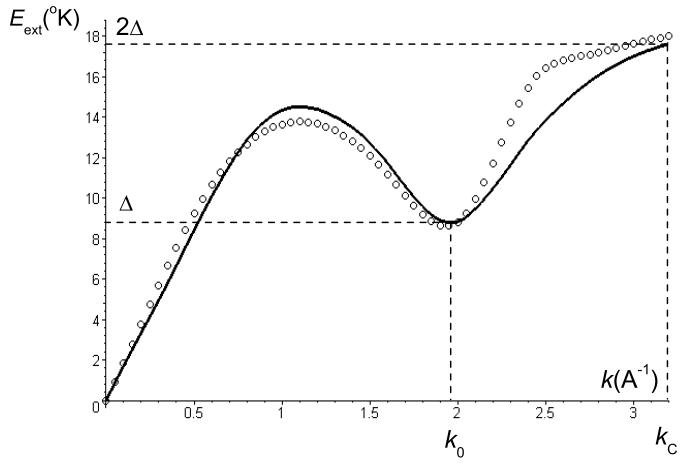


Figure 9: The dispersion curve of CE obtained with the optimal parameter $k_C = 3.2A^{-1}$. As result we have: $\mu = 0.19m$ (the new roton mass), $\Delta = 8.8^{\circ}K$, $k_0 = 1.95A^{-1}$. The dotted line is experimental spectrum of CE [28] at low temperature. In the point $k = k_C$ on the line 2Δ the dispersion curve ends.

We can see in (72) that increase of k_C causes increase of E_0 : $E_0^{(4)} < E_0^{(5)} < E_0^{(6)} < E_0^{(7)}$. Hence the energy $E_0^{(4)}$ is minimal for all allowed energies E_0 . This means that

$$E_0/N = E_0^{(4)}/N = -2.60^{\circ}K. \quad (74)$$

The experimental value is $E_0/N = -7.12^{\circ}K$. It is necessary to notice that short-range correlations make essential contributions in E_0 . These correlations are calculated badly by the collective variables method. So, in [22] it has been proposed the hybrid approach where transition to collective variables was made not completely and part of the coordinates was kept as Cartesian coordinates. Proceeding from aforesaid we can consider the result (74) as satisfactory.

The final spectrum of a collective excitation $E_{ext}(k)$ is shown in Fig.9. The dispersion curve has been obtained as a numerical solution of the set of equations (62) and (64) with the optimal parameter $k_C = 3.2A^{-1}$. We can see

a weak pinning of the dispersion curve to the line 2Δ that means decay of CE into two rotons with energy Δ and momentum k_0 in each. The dispersion curve coincides with the experimental curve well.

B. Suppression of Bose condensate.

In Bose system at sufficiently low temperature the occupation number N_0 of a state with minimal energy (momentum of particles is $\mathbf{p} = 0$) is macroscopic number $N_0 \leq N$. For ideal gas density of a condensate and temperature of the transition are

$$\frac{N_0}{N} = 1 - \left(\frac{T}{T_0} \right)^{3/2}, \quad T_0 = \frac{2\pi\hbar^2}{m} \left(\frac{N/V}{2.612} \right)^{2/3}. \quad (75)$$

At temperature $T = 0$ all atoms are in BC: $N_0 = N$. If particles interact then BC is suppressed. BC is suppressed strongly in HeII: $N_0/N = 0.07$. In the paper [3] the model was proposed where BC is suppressed due to formation of pairs of boson. In this section we shall consider suppression of BC from the standpoint of collective motions of liquid using the oscillator model.

The equations (32) or (69) describe harmonic oscillations of Bose liquid. According to the uncertainty principle the energy of a oscillator in a ground state can't be zero $1/2\hbar\omega \neq 0$. The energy is divisible by kinetic energy and potential energy in equal parts $\langle T \rangle = \langle U \rangle = M\omega^2 \langle \rho_{\mathbf{k}} \rho_{-\mathbf{k}} \rangle / 2 = \xi/2$ according to the virial theorem. This zero point energy is the energy of fluctuations of a quantum liquid. Hence collective motions is in a ground state and no all particles can be in BC. That's BC is suppressed by the dynamical fluctuations [29].

Using the equation (24) let's write the energy of a ground state in the a form:

$$\begin{aligned} & \sum_{\mathbf{k} \neq 0} \varepsilon(k) \left[-4f^2(k) \rho_{\mathbf{k}} \rho_{-\mathbf{k}} - 2f(k) + \frac{1}{4} \rho_{\mathbf{k}} \rho_{-\mathbf{k}} - \frac{1}{2} \right] \\ & + \frac{N^2}{2V} \nu_0 + \sum_{\mathbf{k} \neq 0} \frac{N}{2V} \nu_k (\rho_{\mathbf{k}} \rho_{-\mathbf{k}} - 1) = E_0. \end{aligned} \quad (76)$$

After ground state averaging of this expression according to the rule $\rho_{\mathbf{k}} \rho_{-\mathbf{k}} \rightarrow \langle \rho_{\mathbf{k}} \rho_{-\mathbf{k}} \rangle$, taking into consideration the expression for the potential energy of a system

$$\sum_{1 \leq i < j \leq N} \Phi(|\mathbf{r}_i - \mathbf{r}_j|) = \frac{N^2}{2V} \nu_0 + \sum_{\mathbf{k} \neq 0} \frac{N}{2V} \nu_k (\langle \rho_{\mathbf{k}} \rho_{-\mathbf{k}} \rangle - 1) \quad (77)$$

and the evident equality $E_0 = \langle T \rangle + \langle \Phi \rangle$, we can write the average kinetic energy of a system as

$$\langle T \rangle = \sum_{\mathbf{k} \neq 0} \varepsilon(k) \left[\left(-4f^2(k) + \frac{1}{4} \right) \langle \rho_{\mathbf{k}} \rho_{-\mathbf{k}} \rangle - 2f(k) - \frac{1}{2} \right]. \quad (78)$$

Since $\varepsilon(k)$ is kinetic energy of a particle then the expression in the square brackets can be understood as occupation numbers of above-condensate particles $N_{\mathbf{k} \neq 0}$. Using the connection (29) we can write the kinetic energy and the corresponding occupation number via the function $f(k)$ or via the structure factor $S(k)$:

$$\langle T \rangle = \sum_{\mathbf{k} \neq 0} \frac{\hbar^2 k^2}{2m} \left[\frac{(4f(k) + 1)^2}{-16f(k)} \right], \quad (79)$$

$$N_{\mathbf{k} \neq 0} = \frac{(4f(k) + 1)^2}{-16f(k)} = \frac{(S(k) - 1)^2}{4S(k)}. \quad (80)$$

Then the density of BC is

$$N_0 = N - \sum_{\mathbf{k} \neq 0} N_{\mathbf{k}} = N - \frac{V}{(2\pi)^3} \int_0^\infty \frac{(S - 1)^2}{4S} 4\pi k^2 dk. \quad (81)$$

In the case of non-interacting particles we have: $f(k) = -1/4$ and $S(k) = 1$, hence $N_{\mathbf{k}} = 0$, $N_0 = N$. In the case of weakly non-ideal gas with the interaction $\nu_{\mathbf{k}} = \frac{4\pi\hbar^2}{m}a$ if a scattering length $\frac{a^3}{V/N} \ll 1$ the density of BC is

$$\frac{N_0}{N} = 1 - \frac{8}{3} \sqrt{\frac{Na^3}{\pi V}} \quad (82)$$

that coincides with the result of Bogoliubov [26].

The expression for density of BC (81) is correct for systems with weak interaction only. As it has been shown in [20, 21] for a total case we must use the expression

$$\frac{N_0}{N} = F(R \rightarrow \infty) = \exp \left(-\frac{V}{(2\pi)^3 N} \int_0^\infty N_{\mathbf{k}} 4\pi k^2 dk \right), \quad (83)$$

where $F(\mathbf{r}|\mathbf{r}')$ is a one-particle density matrix, $R = |\mathbf{r}' - \mathbf{r}|$. The expression (81) is the two first terms of an expansion of the exponent (83). In RPA the density of BC is

$$\left(\frac{N_0}{N} \right)_{RPA} = 0.263. \quad (84)$$

In the higher approximations we have the effective Hamiltonian (66). In order to calculate density of BC with the effective Hamiltonian \hat{H}_{eff} (66) we are going to proceed from the following reason. Let's write the energy of a ground state of Bose system in a form:

$$\sum_{\mathbf{k} \neq 0} \tilde{\varepsilon}(k) \left[\frac{(S(k) - 1)^2}{4S(k)} \right] + \frac{N^2}{2V} \nu_0 + \sum_{\mathbf{k} \neq 0} \frac{N}{2V} \tilde{\nu}_k [S(k) - 1] = E_0. \quad (85)$$

We must mark out the kinetics energy of a system of *particles*. All anharmonic terms in the Hamiltonian \hat{H} (10) or in the Hamiltonian \hat{H}_{BZ} (6) are the terms of a operator of kinetic energy $\sum_{1 \leq j \leq N} \frac{\hat{p}_j^2}{2m}$. This means that renormalization of mass $m \rightarrow \tilde{m}$ and interaction $\nu(k) \rightarrow \tilde{\nu}(k)$ occurs due to the contribution of kinetic energy but not the interaction as in usual perturbation theory. Hence the potential energy of a system $\sum_{1 \leq i < j \leq N} \Phi(|\mathbf{r}_i - \mathbf{r}_j|)$ (77) does't change at transition from \hat{H} to \hat{H}_{eff} . In order to obtain the new kinetic energy of particles let's rewrite (85) in the identical form:

$$\begin{aligned} & \sum_{\mathbf{k} \neq 0} \left(\tilde{\varepsilon}(k) \left[\frac{(S(k) - 1)^2}{4S(k)} \right] + \frac{N}{2V} (\tilde{\nu}_k - \nu_k) [S(k) - 1] \right) \\ & + \frac{N^2}{2V} \nu_0 + \sum_{\mathbf{k} \neq 0} \frac{N}{2V} \nu_k [S(k) - 1] = E_0. \end{aligned} \quad (86)$$

Proceeding from aforesaid the first term is a new kinetic energy of particles. In order to obtain a momentum distribution $N_{\mathbf{k}}$ of particles let's mark out kinetic energy of a *particle* $\varepsilon(k)$ in this expression. Then we have

$$\langle T \rangle = \sum_{\mathbf{k} \neq 0} \varepsilon(k) \left(\frac{\tilde{\varepsilon}(k)}{\varepsilon(k)} \left[\frac{(S(k) - 1)^2}{4S(k)} \right] + \frac{N}{2V} \frac{\tilde{\nu}_k - \nu_k}{\varepsilon(k)} [S(k) - 1] \right), \quad (87)$$

$$N_{\mathbf{k} \neq 0} = \frac{\tilde{\varepsilon}(k)}{\varepsilon(k)} \left[\frac{(S(k) - 1)^2}{4S(k)} \right] + \frac{N}{2V} \frac{\tilde{\nu}_k - \nu_k}{\varepsilon(k)} [S(k) - 1]. \quad (88)$$

Substituting (88) in (83) we obtain the densities of BC for the dispersion curves **1-7** in Fig.5:

$$\begin{aligned} N_0^{(1)}/N &= 0.149, & N_0^{(2)}/N &= 0.104, & N_0^{(3)}/N &= 0.071 \\ N_0^{(4)}/N &= \mathbf{0.048} \\ N_0^{(5)}/N &= 0.031 & N_0^{(6)}/N &= 0.020 & N_0^{(7)}/N &= 0.013 \end{aligned} \quad (89)$$

As it has been said before, the curve **4** with $k_C = 3.2A^{-1}$ realizes in practice only. This means that portion of BC in superfluid helium at zeroth temperature is $N_0/N = 0.048$. Density of BC measured in experiment by the method of deep inelastic scattering of neutrons [30] and by the method of quantum evaporation [31] is 0.07. Hence the result (89) is satisfactory fully.

VI. CONCLUSION

As a result of self-consistent solution of Schrodinger equation for the state of Bose liquid with one CE the dispersion law $E_{ext}(k)$ (the set of equations (62) and (64)) has been obtained. It has been shown that for the Hermitian form of Bogoliubov-Zubarev Hamiltonian the second term in the Bijl-Dingle-Jastrow expansion considers real and virtual processes of decay of CE. Higher terms in this expansion considers decays into three and more excitations that is very improbable processes. Decay of CE on two rotons causes existence of the end point k_C of a dispersion curve. Our main result is: *the phonon-roton dispersion curve $E_{ext}(k)$ is determined by both interaction between bosons $\nu(k)$ and the end point k_C . That is the dispersion curve strongly depends on property of its stability.*

In order to find k_C and the energy of ground state we have been formulated the oscillator model of Bose liquid. According to this model the basic B-Z Hamiltonian can be rewritten as Hamiltonian in the harmonic approximation with renormalized kinetic and potential energies. As a result we have a totality of harmonic oscillators again with the same dispersions $S(k)$ but with another frequencies already. Energy of a ground state E_0 depends on these frequencies, hence it depends on the end point k_C . Proceeding from the condition of minimum of energy and from the condition of stability of ground state we can obtain the unknown end point $k_C = 3.2\text{A}^{-1}$ of the spectrum. The obtained spectrum of CE E_{ext} is a function connecting energy of a excitation with the structure factor $S(k)$ of liquid which is taken from the experiment. Thus we did not use any model potentials of interaction. It means that all our calculations have been done without any adaption parameters.

A mechanism of suppression of one-particle BC has been described at temperature of absolute zero. As a consequence of the uncertainty principal the dynamical fluctuations exist in a ground state and give nonzero kinetic energy. These fluctuations suppresses BC. It has been shown that such fluctuations are absent for ideal gas. It is necessary to notice that the analogy dynamical quantum fluctuations in other systems can cause quantum phase transitions [32]. At nonzero temperatures BC is suppressed by both dynamical fluctuations and kinematic (thermal) fluctuations.

Appendix A: The operator of momentum

Let's consider the expression:

$$\begin{aligned}
J^{1/2}\hat{\mathbf{P}}J^{-1/2}\bar{\psi} &= -\sum_{\mathbf{k}_1 \neq 0} \hbar \mathbf{k}_1 \rho_{\mathbf{k}_1} J^{1/2} \frac{\partial}{\partial \rho_{\mathbf{k}_1}} \left[J^{-1/2} \bar{\psi} \right] = \frac{1}{2} \sum_{\mathbf{k}_1 \neq 0} \hbar \mathbf{k}_1 \rho_{\mathbf{k}_1} \frac{\partial \ln J}{\partial \rho_{\mathbf{k}_1}} + \hat{\mathbf{P}}\bar{\psi} \\
&= \frac{1}{2} \sum_{\mathbf{k} \neq 0} \sum_{\mathbf{k}_1 \neq 0} \hbar \mathbf{k}_1 \rho_{\mathbf{k}_1} \frac{\partial}{\partial \rho_{\mathbf{k}_1}} \left[\ln C - \frac{1}{2} \rho_{\mathbf{k}} \rho_{-\mathbf{k}} \right] + \frac{1}{2} \sum_{\mathbf{k} \neq 0} \sum_{\mathbf{q} \neq 0} \sum_{\mathbf{k}_1 \neq 0} \hbar \mathbf{k}_1 \rho_{\mathbf{k}_1} \frac{\partial}{\partial \rho_{\mathbf{k}_1}} \left[\frac{1}{6} \rho_{\mathbf{k}} \rho_{\mathbf{q}} \rho_{-\mathbf{k}} \rho_{-\mathbf{q}} \right] \\
&+ \dots + \hat{\mathbf{P}}\bar{\psi} = -\frac{1}{2} \sum_{\mathbf{k} \neq 0} [\hbar \mathbf{k} \rho_{\mathbf{k}} \rho_{-\mathbf{k}} - \hbar \mathbf{k} \rho_{\mathbf{k}} \rho_{-\mathbf{k}}] + \frac{1}{6} \sum_{\mathbf{k} \neq 0} \sum_{\mathbf{q} \neq 0} [\hbar \mathbf{k} \rho_{\mathbf{k}} \rho_{\mathbf{q}} \rho_{-\mathbf{k}} \rho_{-\mathbf{q}} \\
&+ \hbar(\mathbf{q} - \mathbf{k}) \rho_{\mathbf{k}} \rho_{\mathbf{q} - \mathbf{k}} \rho_{-\mathbf{q}} + \hbar(-\mathbf{q}) \rho_{\mathbf{k}} \rho_{\mathbf{q} - \mathbf{k}} \rho_{-\mathbf{q}}] - \dots + \hat{\mathbf{P}}\bar{\psi} = 0 + \hat{\mathbf{P}}\bar{\psi} \implies \hat{\mathbf{P}}\bar{\psi} = \mathbf{P}\bar{\psi}.
\end{aligned} \tag{A1}$$

That is if the function ψ is eigenfunction of the operator of momentum $\hat{\mathbf{P}}$ then $\bar{\psi}$ is eigenfunction of the operator of momentum too.

Appendix B: The wave function of a system of free bosons

If to exclude interaction between particles: $\nu(k) = 0$, the wave function of a system $\psi = \bar{\psi}J^{-1/2}$ must be a wave function of free bosons. So, we can write for a ground state:

$$\psi = B e^U J^{-1/2} = const \Rightarrow U = \frac{1}{2} \ln J, \tag{B1}$$

where B is a normalization constant. Let's verify it. In a ground state of a system of noninteracting bosons $E_0 = 0$. Then proceeding from the equation (15) we can write:

$$\begin{aligned}
&-\frac{\partial^2 U}{\partial \rho_{\mathbf{k}_1} \partial \rho_{-\mathbf{k}_1}} - \frac{\partial U}{\partial \rho_{\mathbf{k}_1}} \frac{\partial U}{\partial \rho_{-\mathbf{k}_1}} - \frac{1}{4} \rho_{\mathbf{k}_1} \frac{\partial \ln J}{\partial \rho_{\mathbf{k}_1}} - \frac{1}{2} +
\end{aligned} \tag{B2}$$

$$\sum_{\mathbf{k}_2 \neq 0} \frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{k_1^2 \sqrt{N}} \rho_{\mathbf{k}_1 + \mathbf{k}_2} \left[\frac{\partial U}{\partial \rho_{\mathbf{k}_1} \partial \rho_{\mathbf{k}_2}} + \frac{\partial U}{\partial \rho_{\mathbf{k}_1}} \frac{\partial U}{\partial \rho_{\mathbf{k}_2}} \right] = 0.$$

Now let's rewrite the third term of the equation (B2) in the form:

$$\begin{aligned} & -\frac{1}{4} \frac{\partial \ln J}{\partial \rho_{\mathbf{k}_1}} \left[-\frac{\partial \ln J}{\partial \rho_{-\mathbf{k}_1}} + \frac{1}{\sqrt{N}} \sum_{\mathbf{k} \neq 0} \frac{\mathbf{k} \mathbf{k}'}{k^2} \rho_{\mathbf{k} + \mathbf{k}'} \frac{\partial \ln J}{\partial \rho_{\mathbf{k}'}} \right] \\ & = \frac{\partial U}{\partial \rho_{\mathbf{k}_1}} \frac{\partial U}{\partial \rho_{-\mathbf{k}_1}} - \sum_{\mathbf{k}_2 \neq 0} \frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{k_1^2 \sqrt{N}} \rho_{\mathbf{k}_1 + \mathbf{k}_2} \frac{\partial U}{\partial \rho_{\mathbf{k}_1}} \frac{\partial U}{\partial \rho_{\mathbf{k}_2}}, \end{aligned} \quad (\text{B3})$$

where we used the equation for the Jacobian (11) and the equality (B1). After that we have the equation:

$$-\frac{\partial^2 U}{\partial \rho_{\mathbf{k}_1} \partial \rho_{-\mathbf{k}_1}} - \frac{1}{2} + \sum_{\mathbf{k}_2 \neq 0} \frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{k_1^2 \sqrt{N}} \rho_{\mathbf{k}_1 + \mathbf{k}_2} \frac{\partial U}{\partial \rho_{\mathbf{k}_1}} \frac{\partial U}{\partial \rho_{\mathbf{k}_2}} = 0. \quad (\text{B4})$$

With help the relationship (B1) we can write:

$$-\frac{1}{2} \frac{\partial}{\partial \rho_{\mathbf{k}_1}} \left[\rho_{\mathbf{k}_1} + \frac{\partial \ln J}{\partial \rho_{-\mathbf{k}_1}} - \frac{1}{\sqrt{N}} \sum_{\mathbf{k}_1 \neq 0} \frac{\mathbf{k}_1 \mathbf{k}_2}{k_1^2} \rho_{\mathbf{k}_1 + \mathbf{k}_2} \frac{\partial \ln J}{\partial \rho_{\mathbf{k}_2}} \right] = 0, \quad (\text{B5})$$

because the expression in the square brackets coincides with the equation (11). We can see that the wave function (B1) satisfies to the Schrodinger equation (15).

Let's obtain normalization constant B . Let the wave function of a ground state $\bar{\psi}$ (23) is determined by the following way in RPA:

$$\bar{\psi} = A \exp \left[-\frac{1}{4} \sum_{\mathbf{k} \neq 0} \frac{1}{S(k)} \rho_{\mathbf{k}} \rho_{-\mathbf{k}} \right], \quad (\text{B6})$$

where (29) was taken into account. The constant A is obtained from the normalization condition (7) in the thermodynamical limit $N \rightarrow \infty$:

$$\prod'_{\mathbf{k} \neq 0} \int_{-\infty}^{\infty} d\rho_{\mathbf{k}}^c \int_{-\infty}^{\infty} d\rho_{\mathbf{k}}^c |\bar{\psi}|^2 = 1 \Rightarrow A = \prod'_{\mathbf{k} \neq 0} \sqrt{\frac{1}{\pi S(k)}}. \quad (\text{B7})$$

Let's write the Jacobian in RPA

$$J = C \exp \left[-2 \sum'_{\mathbf{k} \neq 0} (\rho_{\mathbf{k}})^2 + (\rho_{\mathbf{k}}^s)^2 \right]. \quad (\text{B8})$$

The constant C is obtained from the condition (5) in the thermodynamical limit $N \rightarrow \infty$:

$$V^N = \prod'_{\mathbf{k} \neq 0} \int_{-\infty}^{\infty} d\rho_{\mathbf{k}}^c \int_{-\infty}^{\infty} d\rho_{\mathbf{k}}^c J \Rightarrow C = V^N \prod'_{\mathbf{k} \neq 0} \frac{1}{\pi}. \quad (\text{B9})$$

Then

$$\psi = J^{-1/2} \bar{\psi} = \frac{1}{\sqrt{V^N}} \prod_{\mathbf{k} \neq 0} \frac{1}{\sqrt{S(k)}} \exp \left(-\frac{1}{4} \sum'_{\mathbf{k} \neq 0} \left[\frac{1}{S(k)} - 1 \right] \rho_{\mathbf{k}} \rho_{-\mathbf{k}} \right). \quad (\text{B10})$$

If interaction between particles is absent, then $S(k) = 1$ and the wave function of a system ψ transforms to the wave function of a system of free bosons $\psi_{\nu=0} = 1/\sqrt{V^N}$, and the normalization constant is

$$B = 1/\sqrt{V^N}. \quad (\text{B11})$$

The constant B is the same for higher powers of expansion of the wave function and Jacobian.

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